SOME HINTS ON THE USE OF LIMITS IN GEOMETRY[1]

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Much has been spoken and written recently concerning the use of limits in elementary geometry, and in some quarters a strong tendency to attempt to abolish the concept from the secondary-school work is manifest. Propositions involving the use of limits are certainly the most difficult with which the secondary teacher has to do. In view of recent discussions it may not be out of place to present here some brief considerations upon the subject, in the hope that the younger teachers may derive some help therefrom.

We may as well admit at the outset that the notion of a limit is indispensable to a clear comprehension of some of the most important things in elementary geometry. The area of a circle is by *definition* the common limit toward which the areas of the inscribed and circumscribed regular polygons approach as the number of sides is increased indefinitely. Similarly, the length of a circumference is by definition the limit of the perimeter toward which the perimeters of these same polygons approach. Analogous statements hold concerning the surface, area, and volume of every solid, whether the boundaries be plane or curved surfaces. In more advanced work the concept enters in practically every case where one has to do with a curved line or a curved surface. The notion of a tangent to a curve is reached through a process of passing to the limit. Examples from physics might also be adduced to show how absolutely indispensable the concept of a limit is. In such cases as have been cited one cannot, properly speaking, say that the question of banishing the notion of a limit is open for discussion.

[1] This article is based upon the recommendations contained in a report of the Committee on Geometry read before the mathematics conference of the Wisconsin State Teachers' Association.

It is impossible to banish the notion without serious detriment to elementary mathematics.

But there is another class of cases that is apt to give the secondary-school teacher much more trouble than any of the cases mentioned above. To this class belong the so-called incommensurable cases in the demonstration of such propositions as "the ratios of the areas of two rectangles having equal bases is equal to the ratio of their altitudes." It is at least an open question whether some modification in the manner of presenting these propositions to beginners may not be desirable.

The only modification that seems feasible consists in confining the demonstrations to the commensurable cases and making a clear statement of the difficulty involved in the incommensurable cases. The writer believes that much can be gained by presenting these cases by means of cases where relations are given by known numbers. Suppose one is presenting the proposition that the ratios of the two pairs of line segments determined by a line parallel to the base of a triangle are equal. The method of procedure might be something as follows:

First step.—Let the pupil actually construct a triangle ABC, the measure of whose side AB is a whole number, say 7. Choose D on AB such that the measure of the distance from A is a whole number, say 4. From D draw a line parallel to BC, cutting AC and E. It is then easy to show that for this particular division of the triangle $AD: DB = AE: EC$, and to generalize to the case where the measures of AB and AD are any whole numbers, m and n.

Second step.—Construct a triangle ABC, for which the measure of AB is a fraction, say 7.7, using the same unit of measurement that was used in the first step. Choose the point AD so that $AD = 4$, then DB is 3.7. The form of proof used in the particular case taken up in the first step cannot be applied, because the segment DB cannot be measured with the unit we have used, but it can be used easily if we take as the unit one-tenth of the original unit. Moreover, *the new unit could have been used in the first step.* The ratio would have taken the form $\frac{40}{30}$, but it is not changed by the change in the unit.

If the lengths had been 7.61 and 4 for *AB* and *AD* respectively, the application of the method employed in the first step would have been impossible without taking a still smaller unit, viz., one one-hundredth of the original unit. And again the new unit could have been used in all previous cases. The larger the denominator of the fraction the smaller the unit of measurement which must be taken. The generalization to the case where the measures of *AB* and *AD* are any fractions $\frac{m}{n}$ and $\frac{p}{q}$ is easy. To find the unit of measure one has only to reduce the fractions $\frac{m}{n}$ and $\frac{p}{q}$ to a common denominator.

Third step.—Choose *AB* and *AD* such that their lengths are incommensurable with each other, say $AB=6+\sqrt{2}$ and $AD=4$. It may be assumed that $\sqrt{2}$ is the measure of the diagonal of the unit square, so that the construction is easy. If we attempt to express $2+\sqrt{2}$ as a decimal fraction the real nature of the difficulty begins to manifest itself. The process of taking a smaller unit of measurement which was employed to overcome the difficulty in the second step will not answer in the present case, for no unit can be found that is small enough. However, one can get a series of triangles which give the approximations,

$$\tfrac{4}{3}, \ \tfrac{40}{31}, \ \tfrac{400}{314}, \ \tfrac{4000}{3141} \ \ldots .$$

corresponding to the units of measurement,

$$1, \ .1, \ .01, \ .001 \ \ldots .$$

and such that less and less remains over of the segment *DB*. The unit used at any stage might have been used in all the preceding stages. The teacher will guard against allowing similar conclusions to be reached when *DB* is measured by a number that is expressed as a repeating decimal.

Having brought the pupil face to face with the difficulty that exists, it is permissible to state clearly that the theorem can be demonstrated rigorously for all cases and that for the time being it will be assumed to be true, whatever may be the measures of the segments *AD* and *DB*.

The foregoing must be looked upon rather as illustration

than as demonstration. It is, except for the notation employed, similar to the incomplete demonstration given in some of the best German texts.

Euclid in his demonstration of the above theorem does not use any limiting process, but the use of incommensurables is implied in his use of the notion of the area of a triangle which is used.[2] English texts following Euclid closely limit the demonstration of the theorem giving the area of a rectangle to cases where the sides of the rectangle are commensurable. That Euclid was familiar with incommensurable lines is amply shown by the tenth book, in which we find no less than 116 propositions devoted to the exposition of the theory of incommensurables. The last proposition of the book is the theorem asserting that the diagonal of a rectangle is incommensurable with a side.

All propositions which involve incommensurable cases may be left thus incomplete, at least on the first reading. If time should permit, the pupil could take up the rigorous demonstration of at least one of these propositions, with profit, after he has gone over the propositions dealing with the area of the circle and the length of the circumference, where the limiting process is much easier to comprehend.

Of course, no teacher would expect to find the demonstration of this theorem written out in the textbook at such length as is here indicated, but it should be borne in mind that *it is not the function of the textbook to take the place of the teacher.*

The actual construction of the figures, which is at best only approximate, is recommended, because it will serve to bring the pupil into closer contact with his problem.

The time of two or three recitations could be gained in the treatment of the problems to determine the area of the rectangle and the volume of the parallelopiped, if the teacher is willing to forego absolute rigor of demonstration. The method pursued in some of the best continental texts, Henrici and Treutlein, for example, may be outlined as follows:

If the measures of the sides of the rectangle are two integers m and n, the demonstration follows at once by dividing the rectangle into m rows

[2] See Euclid, VI, 2.

of squares each containing n squares. If the measures of the sides are fractions say $a=\frac{m}{n}$ and $b=\frac{p}{q}$ the division into unit squares is impossible unless the unit be changed. If N be a common multiple of n and q we can choose for the unit a length which is the Nth part of the original unit. If α and β be the measure of the two sides in terms of the new unit, there are then $\alpha\cdot\beta$ of the small squares and each of these original squares the N^2 times one of the smaller squares. The superficial content of the rectangle is then $\alpha\cdot\beta\cdot\frac{1}{N^2}=\frac{\alpha}{N}\cdot\frac{\beta}{N}=a\cdot b$ as in the first case.

If one or both of the numbers a or b are incommensurable the foregoing work does not apply. We can however choose N large enough so that the unmeasured part of the two sides and consequently the unmeasured part of the rectangle will be smaller than any assigned number, i. e., the error will be arbitrarily small.

Here again the teacher must illustrate by numerical examples and show just what the error is in using commensurable numbers. The pupil may be told that a rigorous demonstration is possible in case the sides are incommensurable. The solution of the problem to find the volume of the parallelopiped will be exactly similar.

This method is of course open to the objection that it does not emphasize the fact that in the last analysis the measured plane area is the ratio of the plane area to a unit square as well as to the objection that it is not rigorous. But it has some conspicuous merits. It is brief, and in the hands of a skilful teacher it is easy to understand; it is absolutely correct as far as it goes and it points the way to the rigorous proof. For these reasons it would seem that teachers should not hesitate to use it if they so desire.

The concept of a limit is an essentially simple concept. However, the definition given in nearly all textbooks on geometry, though it is adequate to meet the needs of elementary mathematics, is faulty, because, for the student who goes into advanced mathematics, it will have to be revised later. The difficulty will be obviated if one defines a limit as *a constant such that the difference between it and a variable which takes a succession of values may become and remain smaller than any pre-assigned number.* The ordinary definition asserts that the variable never reaches its limit, which restriction need not be true. Great care

should be taken to guard against the slovenly way of thinking which allows many pupils to consider that it is sufficient to regard the variable as approaching so near its limit that the difference may be neglected. While this is certainly true so far as any process of actual measurement is concerned, nothing shows more clearly that the pupil has failed to grasp the real significance of the limiting process.

The so-called fundamental theorem of limits which asserts that if variables are always equal, and each of them approaches a limit, their limits are equal, may well be assumed as a postulate. Indeed, to the beginner it seems to be self-evident, since the two variables may usually be regarded as two different expressions for the same thing. This theorem, by the way, is not the fundamental theorem of limits. The term "fundamental theorem" might better be reserved for a theorem which will enable one to establish the existence of a limit. The distinction could better be accorded to the theorem "any variable which never decreases and which always remains less than a constant quantity L, approaches a limit, l, which is less than or at most equal to L." This theorem applies directly to a very great number of cases in elementary geometry. For example, "the perimeter of a regular inscribed polygon never decreases as the number of sides is increased, but it always remains less than the perimeter of any given circumscribed polygon."

Finally, a few words need to be said concerning the real nature of the difficulty that underlies the application of the limiting process to geometry. The difficulty arises as soon as we attempt to compare magnitudes of the same kind and is exactly parallel with the difficulty that arises whenever we attempt to deal with numbers that are incommensurable with unity. In a word, *the difficulty is fundamentally the same as that which exists in the problem of the irrational number in arithmetic.* When one considers that the arithmetical problem did not receive a satisfactory solution until it was taken up by Dedekind, Weierstrass, and Georg Cantor, in the latter half of the nineteenth century, it is small wonder that the high-school pupil finds the geometrical problem difficult.

Since it is through the arithmetic theory of irrationals that the geometric problem finds its easiest and completest solution, it is most important that the teacher have some knowledge of the arithmetic theory. Fortunately, the literature of the subject is now easily accessible in English. Dedekind's original paper has been translated by Professor W. W. Beman and is published by the Open Court Publishing Company, under the title *Essays on the Theory of Numbers.* The essay on "Continuity and Irrational Numbers" is the one which has a bearing on the subject under discussion. The elements of the Cantor theory may be found in Professor Fine's excellent little book on *The Number-System of Algebra,* published in 1891 by Ginn & Co. There is also a good introduction to the subject in Fine's *College Algebra,* also published by Ginn & Co. The section on the "Relation of the Irrational Numbers to Measurement," pp. 65–70, will be of most interest to the teacher of geometry.